Bringing M.C. Escher's *Planaria* to Life

George Hart
Museum of Mathematics
New York, NY USA
E-mail: george@georgehart.com

Abstract

M.C. Escher's lithograph, *Planaria*, embodies mathematical ideas that have inspired me to design building kits that can be used to make structures of octahedra and tetrahedra on a human-size scale. The geometric foundations are outlined and a series of designs are shown that culminate in large constructions made of laser-cut plywood.

Introduction

It is a beautiful fact that regular octahedra and tetrahedra can pack together to fill all space without gaps or overlaps. They can also be assembled as building blocks to make a fascinating variety of geometric structures. The Dutch graphic artist M.C. Escher (1898-1972) was well aware of these mathematical possibilities and used them masterfully in his 1959 lithograph *Planaria* (*Flatworms*), shown as Fig. 1. While many of Escher's other artworks involve optical illusions, nonlinear transformations, or tricks of perspective, *Planaria* is a direct rendering of a possible physical structure, though one based on relatively unfamiliar geometric principles and relationships.

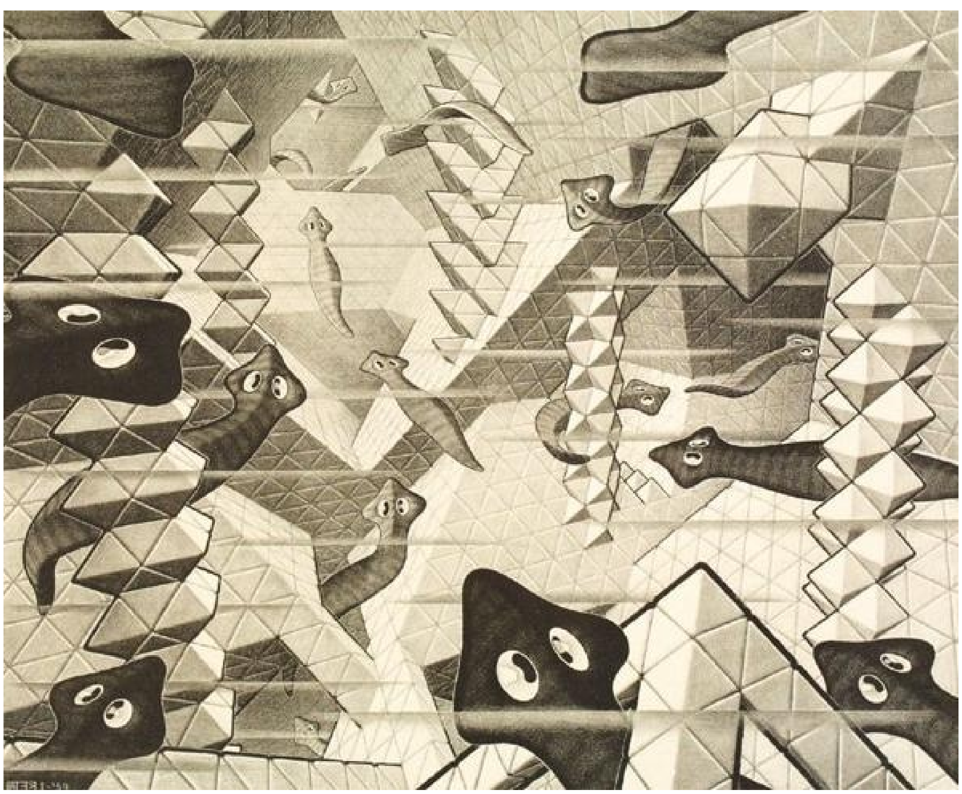

Figure 1: *Planaria, M.C. Escher, 1959*

I have long been fascinated by this image and its rich mathematical foundations, especially the five different parallel columns that penetrate the space. How might one physically replicate the portrayed structure? What Escher is showing initially seems otherworldly to viewers who have grown up in a culture of rectangle-based architecture. Yet it would be literally child's-play—simply stacking blocks together—for anyone who might grow up in a world of octahedral and tetrahedral children's blocks.

Escher clearly had a thorough mastery of the underlying geometric principles in order to create such a rich architectural world. His understanding is also clear in his own written description of the image [2]:

> it is possible to build columns and pillars by piling up tetrahedra and octahedra in such a way that, when viewed as a whole, they do in fact stand vertically. Five of these pillars are shown in the print. The two that stand in the right-hand half of the print are in a sense the reverse of each other. The further to the right of these shows only octahedra, but there must be invisible tetrahedra inside, whereas the pillar to the left of this appears to be built entirely of tetrahedra, yet there must be an internal vertical series of octahedra one on top of the other like beads strung on a necklace.

Escher picked up some geometric ideas from images in books he received from his half-brother, Berend, a mineralogist. His sketch books sometimes show a copy of a textbook geometric image before he made his own variations [5], but I've not seen sketches for *Planaria*. How did Escher obtain his rich understanding of the possible combinations of octahedra and tetrahedra? There are photos of Escher using cardboard and other materials in making physical models as preparation for other examples of his graphic work, but again I've not seen such for *Planaria*. As a sculptor with some sense of the complexities involved, I was certain that Escher must have used physical octahedra and tetrahedra to build his own understanding. So I was not surprised to learn via an email from J.A.F. de Rijk, who knew Escher and wrote (under the pseudonym Bruno Ernst) several books about his work, that in preparation for *Planaria*, "Escher used little forms from clay to make these buildings. And from these buildings he made drawings for his print." So for me to build a deeper tactile understanding, I wanted a kit for creating structures from octahedra and tetrahedra. Moreover, I wanted a kit that is suitable for use when giving a hands-on workshop with students and others. An ideal kit would allow open-ended play with these shapes, but what should that entail, exactly? There are several options for how we might allow the components to interconnect.

Geometric Understanding

There are two natural approaches to explain the octahedron/tetrahedron structure: start with a rhombohedron or start by packing spheres [1]. Six congruent rhombi can be joined to make the rhombohedron of Fig 2. If we choose a rhombus in which the acute angles are 60 degrees, then we can imagine adding the six short diagonals that bisect each rhombus into two equilateral triangles. And then it is easy to see that the rhombohedron is dissected into a regular octahedron and two regular tetrahedra. Next, think of the rhombohedron as a skewed cube, and just as cubes pack all space, it is easy to see how the rhombohedron packs all space without gaps or overlaps. And via the dissection, this implies that regular octahedra and tetrahedra pack all space. A careful examination of this construction shows that each octahedron is surrounded by eight tetrahedra and each tetrahedron is surrounded by four octahedra.

To summarize the above construction: each block joins face-to-face with blocks of the other type. This structure is sometimes abbreviated as the "oct-tet" packing. Another way to understand it starts with piles of cannonballs or spherical fruit. Fig. 3 shows a square pyramid of balls, starting with a square array as the bottom layer and adding successive square layers, each one size smaller and offset to sit in the hollows of the layer below. Each ball is tangent to twelve others: four below it (on which it sits), four in its layer, and four above it. If we construct line segments that connect the center of each ball to the centers of the twelve adjacent balls, the segments outline tetrahedra and octahedra. A close examination reveals that this process creates the same oct-tet structure and it can be extended to fill all space.

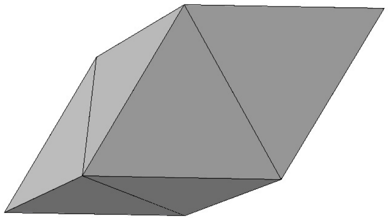

Figure 2: *Rhombohedron = 1 octahedron + 2 tetrahedra.*

Figure 3: *Balls stacked with square base.*

Figure 4: *Balls stacked with triangular base.*

The oct-tet structure can also be generated from a triangular pyramid of balls, as in Fig. 4. The easiest way to see this is to imagine taking the square pyramid, picking it up, rotating it, and putting it down so any of its four triangular faces is on the bottom. Now each horizontal layer is a triangular array of balls. The twelve neighbors can be accounted for as three in the layer below (on which it sits), six in its layer, and three in the layer above. It is a good exercise to look at *Planaria* and find both types of pyramid. (When making the triangular pyramid, there is no ambiguity about where to place each ball, because the pyramid edges serve as guides. But if one has unbounded planes of triangularly arranged balls, one must be sure to have each ball directly over a ball in the layer *three* below it. If instead each ball is over a ball *two* layers below it, a different structure results, called the "hexagonal close packing".)

Because the oct-tet structure is highly triangulated, it has great strength, so architects and engineers find it useful for spanning wide areas. Buckminster Fuller popularized this usage and coined the name "octet truss" for it. Look upward next time you are in a mall or other large open indoor area and you are likely to find the oct-tet structure supporting the ceiling. There are two natural orientations for this structure, which correspond to the square-based and triangle-based pyramidal ball packings. In the first case, a four-fold rotation axis is vertical, while in the latter case a three-fold rotational axis is vertical.

Magnetic Blocks

One direction that I explored is to make 6 cm magnetic octahedra and tetrahedra blocks with a rapid prototyping machine. I made two versions on this idea, shown in Figures 5 and 6. In the first version, there are three small magnets in each face. Magnetic North is facing outwards in the octahedra, while South faces out in the tetrahedra. (Or maybe it is the reverse of that.) This naturally leads users to maintain the constraint that each type of block is surrounded by the other. Having three small magnets provides rotational alignment, i.e., the mating triangles of adjacent blocks remain parallel.

In a second magnetic block design, I placed a spherical magnet in the center of each face. These are confined to a small space in which they are free to rotate. In this design, any block will stick to any other block, since the magnets automatically rotate so North faces South. This allows a wide range of irregular constructions, many of which do not fill space. For example, one can try to put five tetrahedra together around an edge and discover that they almost close. There is a small gap because the dihedral angle of the tetrahedron is slightly less than $2\pi/5$. And similarly, twenty regular tetrahedra can be placed around a point, but there is still some remaining space, so they do not form an icosahedron. See Fig. 7. Having one magnet per face instead of three allows free rotation about the face centers, which has pros and cons. It opens up a larger space of possible structures, but one must align blocks to make oct-tet structure. Fig. 8 shows a part of the tower one-from-the-right in *Planaria*. (It is a stack of *stella octangulas*.)

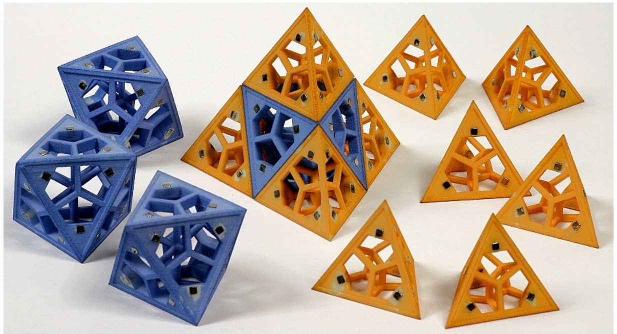

Figure 5: *Blocks with 3 magnets per face.*

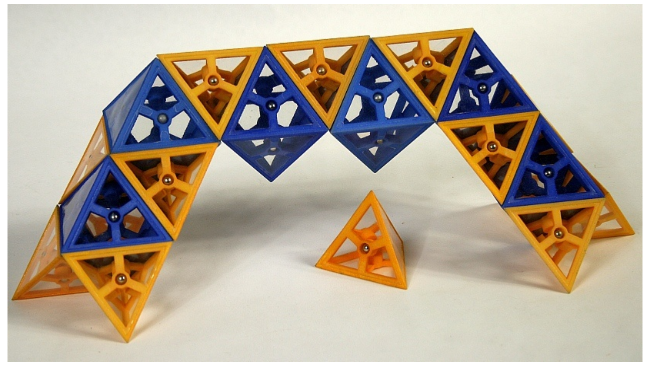

Figure 6: *Blocks with 1 spherical magnet per face.*

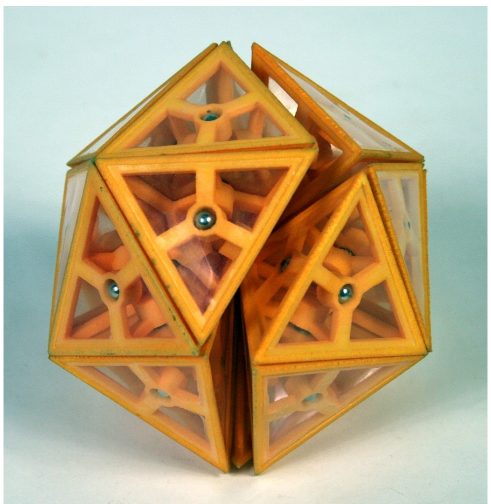

Figure 7: *Not an icosahedron*

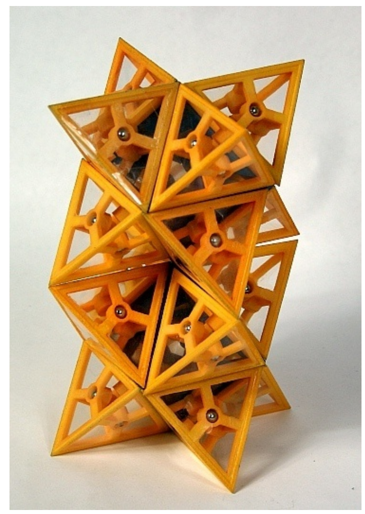

Figure 8: *Tower from Planaria*

So far, I have only made these blocks in small quantities on a rapid prototyping machine. I dye the nylon blue and yellow, then the magnetic spheres are inserted, and a clear cover is glued over each face to retain the magnet but allow rotation. I would love to have enough to make larger constructions, but they are expensive to produce in this manner. So I have also explored other ideas.

Experimental Prototypes of Node and Strut Systems

Instead of making octahedra and tetrahedra as the fundamental building blocks, another strategy is to make nodes and struts that serve as the vertices and edges of the structure. Each node needs twelve connection points in the directions corresponding to twelve adjacent cannonballs. With a bit of sketching, you should convince yourself that these turn out to be the twelve directions from the center of a cube to its twelve edge midpoints. Zometool green struts [3] or the Synestructics purple struts [4] also point in these directions, so can also make these same structures, but on a much smaller scale than I wanted.

There are many ways to construct a connector with receptacles or prongs pointing in these twelve directions and I have experimented with a series of node and strut systems. I am interested in kits with large struts, which can be used by groups of people to make structures on a human scale. In very large quantities, injection molded plastic would be an option, but that is not cost effective in small quantities. So I have explored ways to construct kits from easily available materials. The following designs were instructive, but all failed for one reason or another:

A. The nodes are cubes of wood. A pair of holes is drilled in each edge. The struts are pairs of dowels spaced to match the holes. The advantages of this system are that wood cubes are relatively inexpensive mass-produced parts, holes are easy to drill, and the pairs of dowels provide rotational alignment. However, prototyping showed that at the desired scale, the wood cubes are too heavy.

B. The nodes are cubes of high-density foam with one hole in each edge and the struts are dowels covered in a foam sleeve (commercially made for swimming). See Fig. 9. A notch in the end of the sleeve mates to the angle of the cube to provide rotational alignment. This solves the node weight problem, but the foams I tried were either not rigid enough or not resilient enough. The holes would stretch permanently.

C. The nodes are foam cubes with North magnets at the edge midpoints. The struts are dowels with South magnets at their ends. See Fig. 10. After experimenting, shattering some neodymium magnets, and eventually finding magnets that do not shatter, this design turned out to be too expensive to pursue.

D. The nodes are made of twelve pieces of dowel mitered to join together and the struts are hollow foam tubes. See Fig. 11. This worked, as the tetrahedron in Fig. 12 shows. But these prototypes were time consuming to make, so I had a plastics fabricator cast one of the wood nodes so it could be replicated in plastic. A variety of plastics were tried, but the nodes were always too heavy for the struts to support. In addition, we were concerned that the mace-like appearance of a node in a strut might invite misbehavior.

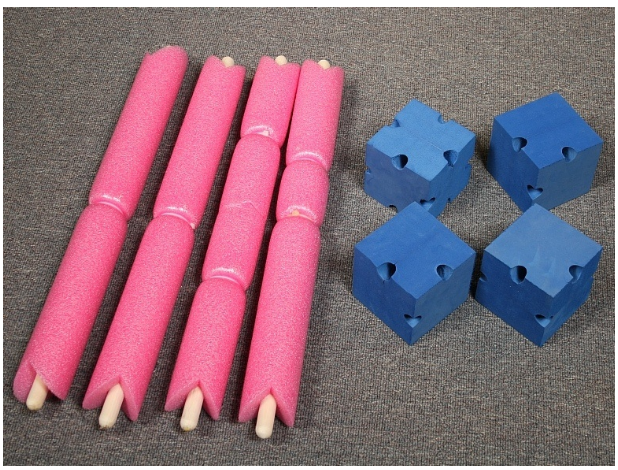

Figure 9: *Foam nodes; dowels for struts.*

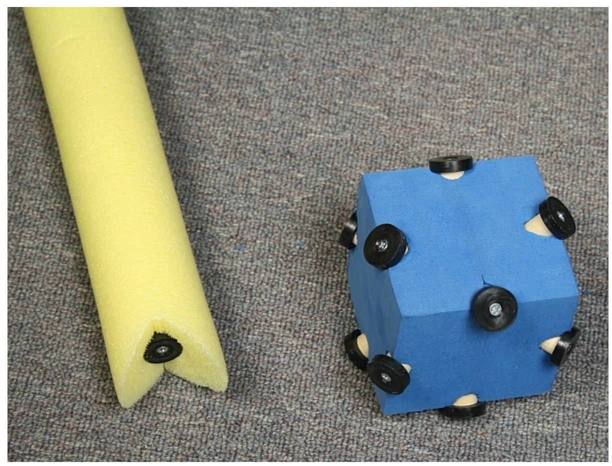

Figure 10: *Magnets in nodes and struts.*

Figure 11: *Wood dowel nodes; foam struts.*

Figure 12: *A tetrahedron*

Large Laser-Cut Wood System

After the above experiments, and after the purchase of a laser-cutter for use in prototyping, I developed a series of designs in which the nodes and struts are made of laser-cut 6 mm thick plywood. As Fig. 13 shows, the construction allows large models, for human-scale constructions, which I like because building anything big is memorable. But large kits are problematic because if a structure looks strong enough to climb on, children will attempt to do so. Then parts will break if they are too weak or the climber may fall off if the structure holds. The approach taken here is to make parts that are clearly too thin to support climbing. Another issue is that long struts can act as lever arms, placing large torques on the nodes, which is mitigated here with multi-piece nodes that pop apart when overstressed.

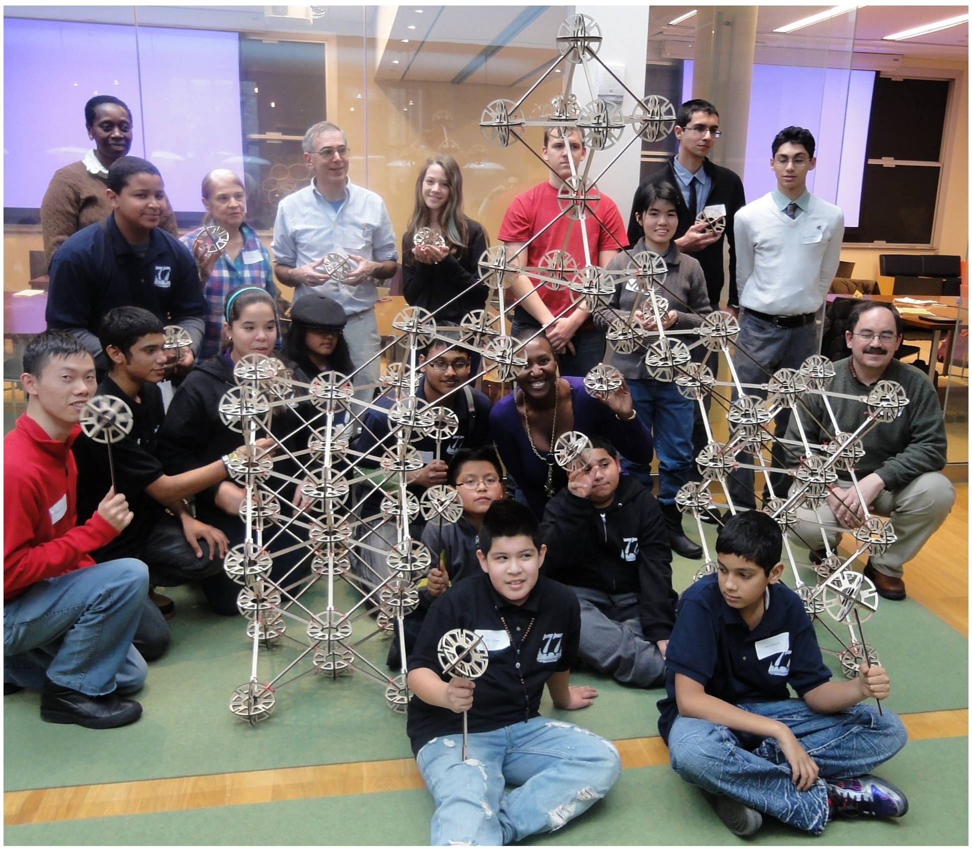

Figure 13: *Construction with laser-cut wood nodes and struts.*

To keep the nodes parallel, the struts should connect in a way that prevents them from rotating. This is easy to arrange with struts cut from flat stock. But using 2D stock to make 3D nodes to which the struts connect and point in the correct directions is trickier. Recall from Figs. 9-10 that the struts must connect in the twelve directions that correspond to the midpoints of the edges of a cube. Given that a laser-cutter produces flat pieces, it is natural to think of assembling the nodes using some sort of polyhedral design. Four possible ways to do this are shown in Figures 14-17. In each case, the same twelve spatial directions are produced. One idea is to build a node that is essentially a cube with connection joints at its edge midpoints. This is feasible with six flat components used as faces of the cube, mimicking design (A) above but lighter in weight. Another family of designs is based on a rhombic dodecahedron, which has twelve face planes orthogonal to the desired directions, so a receptacle in the center of each face could receive a strut. It is also possible to design a node using a cuboctahedron, as its twelve vertices point in the desired directions. This is intriguing also because the edges of a cuboctahedron outline four hexagons, so one can imagine building it from four interlocking laser-cut components.

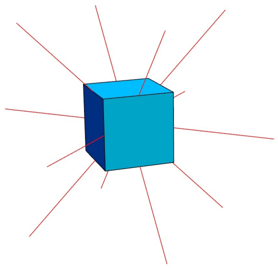

Figure 14: *Possible cube node.*

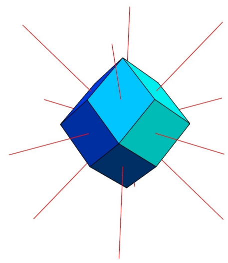

Figure 15: *Possible rhombic dodecahedron node.*

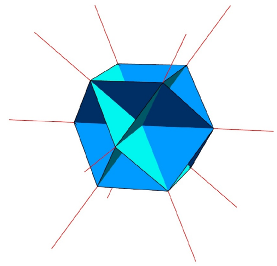

Figure 16: *Possible cuboctahedron node.*

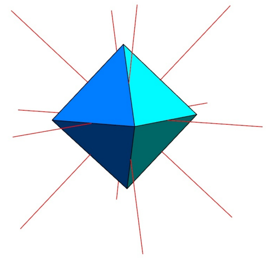

Figure 17: *Possible octahedron node.*

However, the design I ended up producing is based on an octahedron. The octahedron's edge midpoints are equivalent to the cube's edge midpoints, because a cube and an octahedron are mutually dual and can be positioned so their edges bisect each other. Rather than making the eight triangular faces of an octahedron as in Fig. 17, I made a spherical model of the edges. After a number of iterations, I worked out a design with six D-shaped pieces that snap together to make the 12 cm spherical node shown in Fig. 18. The part shape, shown in Fig. 19, has two tabs that lock into holes in another piece. It is something of a puzzle to assemble the nodes, but once built, they hold together without glue, or can be glued for a more solid feel. But an advantage to not gluing them is that they can then pop apart if under extreme stress, rather than breaking the wood. Several cutouts in the parts keep the weight down and also make the nodes visually light and interesting. They are simple, strong, relatively inexpensive, and the most elegant oct-tet design I have made yet, so I had an industrial laser-cutting firm produce 300 nodes and 700 struts.

One issue to be aware of with this design is that the struts do not support any tension. They slide off a node as easily as they slide on. It is hard to laser-cut plywood parts for a good friction fit, because it would require stock of a very consistent thickness, which is not feasible as the wood is sanded after laser-cutting to remove smoke marks. Instead, I designed each strut with a small hole near each end and a twist-tie is placed through the hole to tie the strut to its node. This is usually only needed on the horizontal struts, as the vertical struts are generally under compression.

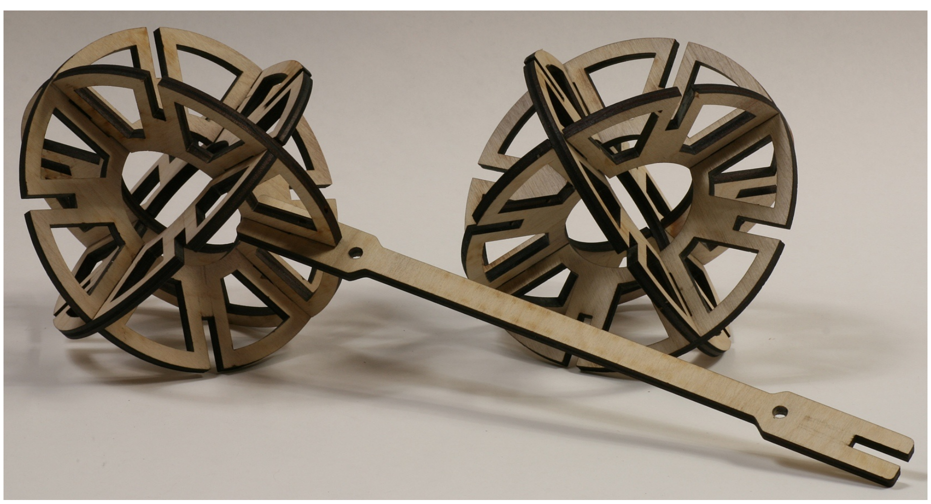

Figure 18: *Laser-cut nodes and strut.*

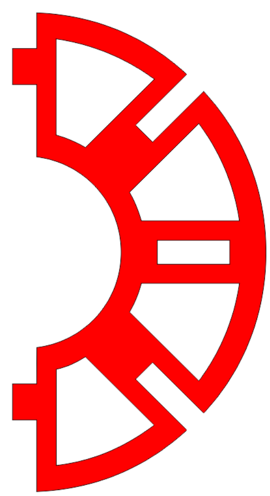

Figure 19: *Template*

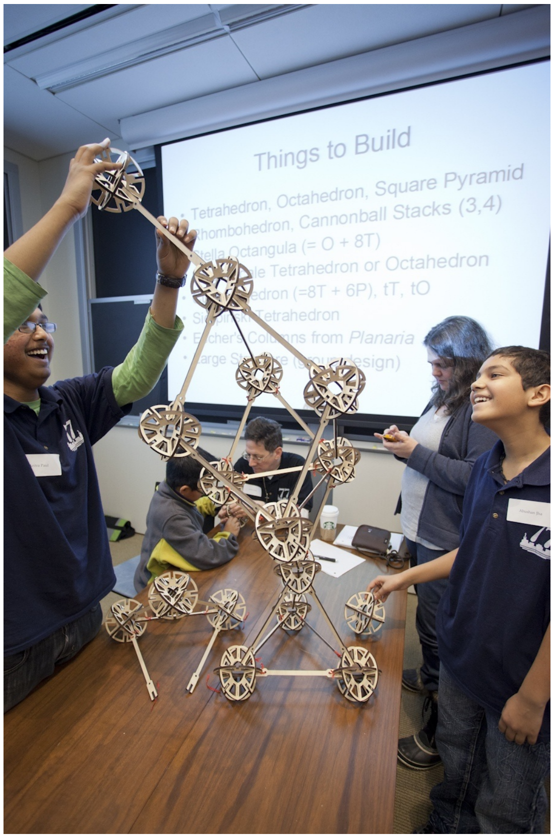

Figure 20: *Workshop participants exploring.*

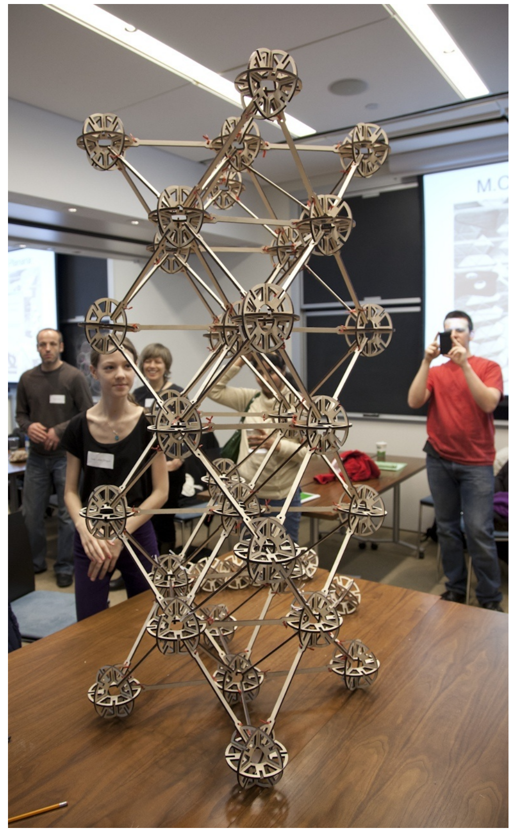

Figure 21: *A tower from Planaria.*

In January, 2012, I ran a workshop at *Math for America* in Manhattan where teachers and students first beta-tested these units with me. Figs. 13 and 20-21 show some constructions with this kit, including one of the towers from Escher's *Planaria*. Additional photos can be seen online at `georgehart.com`

Conclusion

Among the many works of M.C. Escher that have provided mathematical inspiration to his fans, *Planaria* stands out to me for inviting large-scale physical constructions. The assembly of octahedra and tetrahedra into architectural structures is made natural when using kits designed with the appropriate angles and connectors. Small magnetic building blocks provide one level of understanding, while a large node-and-strut kit allows an immersive human-scale experience. After a long prototyping phase, I have worked out a design with laser-cut plywood components suitable for group workshops. Initial testing shows that human-scale construction results in great student impact. Through future workshops, I hope more people will be able to appreciate Escher's wonderful structural visions.

References

[1] H.S.M. Coxeter, *Introduction to Geometry*, 2nd ed., Wiley, 1969.
[2] Bruno Ernst (pseud. for J.A.F. de Rijk), *The Magic Mirror of M.C. Escher,* Ballantine, 1976.
[3] George W. Hart and Henri Picciotto, *Zome Geometry,* Key Curriculum Press, 2001.
[4] Peter Pearce, *Structure in Nature is a Strategy for Design*, MIT, 1978.
[5] Doris Schattschneider, *Visions of Symmetry,* Freeman, 1990, p. 247.