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Goldberg Variations

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1. Introduction and Small Examples

Certain families of polyhedra are worthy of special mention because of their history, applications, beauty, and/or mathematical properties. The *Goldberg Polyhedra* are notable in all these ways. I will introduce them with simple examples, examine some of their properties, and illustrate a range of geometric sculptures that they have inspired me to create. A typical example is shown in Figure 0. It is called GP(3,2) where the 3 and 2 indicate the structure of a path from any pentagon to a neighboring pentagon. Stand on any pentagon, face outwards in any of the five directions, walk three steps away, then turn right 60 degrees, walk two more steps, and you always land on a neighboring pentagon.

Fig. 0. *A typical Goldberg Polyhedron, called "GP(3,2)," with a pentagon-to-pentagon walk highlighted to indicate the meaning of the 3 and 2 in the name.*

Definition: *Goldberg Polyhedra* are the polyhedra that have: (1) pentagons and hexagons for faces, (2) trivalent vertices, i.e., three faces meet at each vertex, and (3) icosahedral symmetry.

The family provides a range of design options for many practical problems and the forms are beautiful just in themselves. A Goldberg polyhedron may have from a dozen faces to an arbitrarily large number. The two simplest forms in this class can be traced back at least 2000 years to classical Greek mathematics, but as a group these shapes have been identified and studied only since the twentieth century. The credit for the geometric insight and formal definition of this family goes to the mathematician Michael Goldberg, who published the definitive paper about them in the 1930's. As with many mathematical ideas, these started out as the imaginings of a pure mathematician exploring an abstract puzzle that interested him. Only years later did it turn out that physical applications for the mathematics were found. Goldberg probably did not surmise that the polyhedra he wrote about would be found, for example, in subatomic particle detectors, loudspeaker design, virus macromolecules, or carbon chemistry. Other applications include architecture, spherical game boards, golf ball dimple patterns, pave jewelry, cartography, and abstract artwork.

These polyhedra are similar to spherical patterns of packed living cells, though more regular. So they have an organic quality that resonates with our sensibility for natural structure. They remind us of various microscopic organisms, plants' seed pods, and skin pattern textures. In my sculpture, I seek to make forms which are simultaneously mathematical and organic [17]. So I find these polyhedra to be a natural foundation for design, as illustrated below.

As there is no standard name for these forms, I now propose to belatedly name them *Goldberg Polyhedra.* For brevity, I will refer to them generically as *GP*, and identify specific examples with a pair of numbers, GP(*a,b*), which indicate the type of pentagon-to-pentagon "60-degree knights move" indicated in Figure 0. The best way to understand the family is to start looking at the simplest possible examples, which turn out to have 12, 32, 42, 72, 92, 122, … faces. What is the pattern in the sequence of numbers? Why does the number of faces always end in a "2" and why are there none with 22, 52, 62, or 82 faces? These issues are discussed in Section 2. First I introduce the family and some of its characteristics by considering particular examples.

The best way to understand three-dimensional geometry is to make physical models. Templates for making paper versions of some of these simple examples are given in the Appendix. The reader is encouraged to cut the face shapes from card stock and tape them together into a 3D model to get a fuller understanding of the forms.

12 Faces, GP(1,0), The Regular Dodecahedron. We will see below that every GP contains exactly twelve pentagons. They can be designed with an arbitrarily large number of hexagons, but there are always exactly twelve pentagons. So the simplest case is to have just the necessary 12 pentagons but no hexagons. The result is the dodecahedron, shown in Figure 1. This is an ancient form studied in depth by classical Greek mathematicians, but known much earlier. Figure 1 is a drawing by Leonardo da Vinci, in a “solid edge” style he invented. It clearly presents both the front and rear surfaces by giving solidity to the edges and opening up each face. This style is useful for both educational and artistic purposes and is easy to implement in software, so is used often below [15].

Fig. 1: *Leonardo’s drawing of the regular dodecahedron for Luca Pacioli’s 1509 book, The Divine Proportion. In the context of this paper, the dodecahedron is seen as GP(1,0). It is the limiting GP case with zero hexagons.*

In terms of the first property in the definition above—pentagons and hexagons—this example makes clear that hexagons are not required. The definition should be understood to mean that anything other than pentagons and hexagons is excluded. All the other GPs include hexagons, so the dodecahedron might seem too simple to group with the family, but in mathematics it is always good to start with zero and to be inclusive.

The third clause in the definition—icosahedral symmetry—is illustrated in Figure 2, which shows the symmetry axes of the dodecahedron. Any line passing through the center of a face, through the center of the dodecahedron, and continuing out the center of the opposite face is a 5-fold axis of rotation. If the dodecahedron is rotated about this axis for a fifth of a revolution, it appears unchanged. As there are twelve faces, arranged in six opposite pairs, there are a total of six 5-fold axis lines. The lines from one vertex through the center and out the opposite vertex are 3-fold axes of rotation. There are twenty vertices, so there are ten 3-fold axes. Any line from an edge midpoint, through the center and out the opposite edge midpoint is a 2-fold axis. The thirty edges give fifteen 2-fold axes. The form appears unchanged when rotated one n^{th} of a revolution about any of these thirty-one n-fold axes. If we imagine deleting the dodecahedron from Figure 2 and just keeping this pattern of axes, each labeled somehow with its order, we will have a representation of what is meant by “icosahedral symmetry.” If start with any of the GP polyhedra below, draw all their axes of rotation, delete the polyhedron, and just look at the geometry of the axes of rotations that keep the object appearing unchanged, we get this exact same arrangement of thirty-one lines in space.

Fig. 2: *Regular dodecahedron, i.e., GP(1,0), with its symmetry axes.*

32 Faces, GP(1,1), The Truncated Icosahedron. Our name for the second in the series, the well known 32-faced form shown in Figure 3, is GP(1,1). To mathematicians, the shape is the "truncated icosahedron," a name which has been definitive at least since the early 1600's, when the astronomer and mathematician Johannes Kepler wrote of it. But the form was well-known to earlier artists and mathematicians in the Renaissance, and Archimedes had already written about it before 200BC. Fig. 3 shows Leonardo da Vinci’s solid-edge drawing of it. Its design is most familiar in what is called a "soccer ball" in the United States or "football" in Europe. The GP soccer ball was invented in the 1950s in Denmark and the familiar black and white version became official starting with the 1970 world cup. [23] The shape has also received considerable attention since the Nobel prize winning discovery in 1985 that carbon atoms naturally join into structures which chemists refer to as "C60." The 60 refers to the number of 3-way joints, which are the positions where carbon atoms sit. To the chemist, these 60 parts are the building blocks of the molecule, so it is natural for them to give it a name that involves the number 60. But if one is physically building the polyhedral form out of hexagonal and pentagonal components, then the number 32 is more important, as it is how many faces one must have on hand.

Fig. 3 *GP(1,1) is our name for the truncated icosahedron. Leonardo’s drawing of it.*

Twelve of the faces are pentagons and the remaining twenty are hexagons. To understand its derivation as a "truncated icosahedron," consider the icosahedron, which has twenty equilateral triangular faces, shown in Figure 4. "Truncation" means to cut off the corners. This reveals a pentagon under each cut and leaves a hexagon from each original triangle. Because our definition's clause 3 requires symmetry, we must cut all the corners to the same depth. (If we didn't, somewhere there would be two adjacent corners with different depths, and a half turn about the edge midpoint between them would swap the two depths, so would not leave the form appearing unchanged.) We can, however, choose the depth of truncation, so there is a family of GP(1,1) forms with different shapes for the hexagons. The particular truncation depth which results in regular hexagons is very attractive, so I use it as a representative form. But it is important to understand that each GP represents a family of geometric variants. More choices among the variations are discussed below.

Fig. 4 *Icosahedron, and truncation process to generate various forms of GP(1,1).*

42 Faces, GP(2,0), The Truncated Rhombic Triacontahedron. The best way to get familiar with geometric forms is to make physical models. Figure 5 shows a model of GP(2,0) constructed from paper polygons and Scotch tape. To make your own, cut out 30 irregular hexagons and 12 pentagons using the template in the Appendix and tape together as shown. Because the vertices are trivalent, you do not need to worry about the dihedral angles between the face planes. The hinge of the tape joint automatically ends up at the correct angle. But it takes careful attention to produce the correct topology and have all the hexagons rotated correctly.

Fig. 5 *Five inch paper model of GP(2,0) colored dark and light like a super-soccer ball.*

When most people first see this, they assume that it is the familiar soccer ball shape. That is natural because one sees a pattern of pentagons separated by hexagons, and the increase from 32 to 42 faces is not enough to jar familiar expectations. As far as I know, the GP(2,0) shape has not been used for any athletic spheres, but it would be wonderful if a new sport were created which used it. That would lead players to sharpen their geometric discrimination in order to know which ball is which.

Of course one big difference between GP(1,1) and GP(2,0) is in the number of faces each has, but with some study, one becomes attuned to two other differences, to discriminate between them at a glance. The first difference is that in GP(2,0) there are some corners where three hexagons (and no pentagons) meet, but in GP(1,1) every corner has exactly one pentagon. So a GP(2,0) sports ball would have some all white corners, while the familiar soccer ball has black touching every corner. A second difference is that the twenty hexagons in GP(1,1) are regular, meaning the angles are all equal and the lengths of the edges are all equal. But a close look at GP(2,2)'s thirty hexagons shows they are not regular.

The GP(2,0) form can be derived from the rhombic triacontahedron, a polyhedron with thirty rhombic faces, shown in Figure 6. Notice it has both 5-fold and 3-fold vertices. By truncating all the 5-fold vertices, each rhombus is reduced to just a central hexagon. Again, we have a choice of how deep to cut and a natural choice is the depth which results in equilateral hexagons, but they will not be equiangular. The obtuse angles in the rhombus, which meet in groups of three, are about 116.5 degrees and these angles remain unchanged by truncation at the acute angles. Thus the GP(2,0) hexagon differs from a regular hexagon, which has all 120-degree angles. In fact, if we did try to put three regular hexagons at a vertex, the face angles would sum to 360 degrees, which is flat, so the form will not curve to make a ball. For this reason, we must find many irregular hexagons in all the GP forms below, which all have vertices with three hexagons, but not all the hexagons need be irregular.

Fig. 6 *Rhombic triacontahedron is truncated to produce GP(2,0).*

72 Faces, GP(2,1), The Truncated Pentagonal Hexecontahedron. Figure 7 shows two mirror-image forms of GP(2,1). This is the simplest GP which is chiral, meaning it has distinct left-hand and right-hand forms. The three above are reflexible, meaning they appear unchanged when reflected in a mirror. The lack of mirror symmetry makes this form more interesting visually. One naturally spends more time studying it and understanding its patterns. Below, when considering more complex GPs, most of them will be chiral and only certain special cases will be reflexible.

Fig. 7 *GP(2,1) consists of twelve pentagons and sixty congruent hexagons. It comes in two enantiomorphic forms.*

Figure 8 shows one way to derive GP(2,1). Start with the pentagonal hexecontahedron, which is a well known Catalan polyhedron (to those who well-know it). It is the dual of the Archimedean snub dodecahedron and has sixty congruent 5-sided faces. Truncate its twelve 5-fold vertices as shown, to create twelve pentagons and sixty hexagons. It is at this step, I feel, that we leave the realm of shapes that have good classical descriptions and move into territory which is best described in the terms "GP(*a,b*)." Knowledge of this family provides a storehouse of useful forms that should be in the mental inventory of any geometric designer. For example, physicists designing the Spin Spectrometer at the Oak Ridge Heavy Ion Research Facility needed a way to arrange a sensor array spherically around a target, to capture a 3D distribution of particles emitted in all directions. They could afford about six dozen sensors and each covers a roughly equal roughly circular area. The solution they chose is the geometry of pentagonal and hexagonal particle detectors spherically arranged with the structure of GP(2,1). [21] In another application, one manufacturer has recently produced a wiffle ball of this shape, claiming its aerodynamic properties make it take a curved path when thrown.

Fig. 8 *GP(2,1) can be derived by truncating the 5-fold vertices of the Pentagonal Hexecontahedron.*

An art application of GP(2,1) is that is the foundation for a computer image "Puzzle" which I made in 1995, shown in Figure 9. As my write-up at the time states, the first part of the puzzle is to figure out the underlying polyhedron, because I discovered that casual viewers assumed it was a soccer ball. More recently, I made the hollow wood sculpture shown in Figure 10. At five inches in diameter, it is an excellent size to hold and spin in ones hand while appreciating the form. To make it, I carefully beveled 72 laser-cut wood parts, giving them the appropriate dihedral angles, then epoxied them edge to edge.

Fig. 9 *"Puzzle" digital image, 1995*
Fig. 10 *"GP(2,1)" aspen, 5 inch, 2003*

One interesting property that the model in Figure 10 makes very clear is that the hexagons are not parallel to opposite hexagons. When you rest it on a horizontal table with a hexagon in contact with the surface, no face is horizontal on the top. Some reflection shows that there is no reason to expect each face in a polyhedron to have an opposite parallel face, even if it is symmetric. For example, among the five Platonic solids, the tetrahedron also has this no-parallel-face property. The hexagons of GP(1,1) and (2,0) are orthogonal to 3-fold and 2-fold symmetry axes respectively, so they do come in opposite parallel pairs. And all pentagons in all GPs are orthogonal to 5-fold axes, so always have an opposite parallel partner. But it can be mildly surprising to see how the GP hexagons in general need not have a parallel partner. Icosahedral symmetry does not imply central symmetry.

92 Faces, GP(3,0), Dual to a three-frequency icosahedron. The next example, GP(3,0) in Figure 11, is best explained in terms of duality and geodesic domes. Buckminster Fuller popularized dome structures that are derived from an icosahedron, where each triangular face is divided into an array of smaller triangles, and then the vertices are projected to a sphere. If each icosahedron edge is divided into k equal parts, he called the result a "k-frequency icosahedron". Figure 12 shows the derivation of a 3-frequency icosahedral sphere. Interestingly, when each edge is divided into k parts, the triangle is divided into exactly k^2 smaller parts, e.g., 9 parts in this 3-frequency example. This is easy to see in general if one understands that area always scales as the square of the length scale factor. If the large triangle is scaled by $1/k$ to produce a small triangle, the area becomes $1/k^2$ of the original area, so k^2 of them are required to fill the large triangle. (Thus a pedant would be correct in always annoyingly saying "k triangled" in place of "k squared.") In the projection step, the vertices are adjusted radially to have unit length, giving a "k-frequency icosahedral geodesic sphere." Then a dome, of course, is just half or some other fraction of the sphere.

The GP(3,0) form results from constructing the dual to the 3-frequency icosahedron. The dual operation is best understood in the 2D case first. Figure 13 shows a triangular tessellation and its dual hexagonal tessellation. Given the triangle pattern, to generate its dual one constructs a point in the center of each face and then connects pairs of points that lie in adjacent faces. The same procedure generates the triangular tessellation if one starts with the hexagonal one, so the dual is an operation of order two, i.e., $dual(dual(x)) = x$. In the 3D case, the situation is analogous but there are geometric variants which might each be called "a dual". It should be clear from Figure 14 how the 3D dual follows the pattern of the 2D dual.

Fig. 11 *GP(3,0) consists of twelve pentagons and eighty hexagons. It is reflexible.*
Fig. 12 *Derivation of the 3-frequency icosahedron: (a) underlying icosahedron, (b) edges divided into thirds, (c) projection to sphere.*
Fig. 13 *Duality in the plane. The hexagonal tessellation and triangular tessellations are dual to each other. Each has vertices in the centers of the faces of the other. The edges of each cross the edges of the other at right angles.*
Fig. 14 *Duality of 3-frequency icosahedron and GP(3,0), looking down on just one hemisphere.*

One issue that comes up here is whether the faces are planar. By definition, polyhedra have planar faces. Any three points in space form a triangle, which must lie in some plane, so there is no question of planarity in triangulated structures such as the *k*-frequency icosahedral sphere. But given the five or six points defining a face of some GP, we need to be sure they lie in a plane if the form is truly a polyhedron. As we will see below, the definition of duality in 3D guarantees planarity in this construction.

This is a good point to mention differences between GPs and "Fullerenes," the spherical allotropes of carbon that are also called "Buckminsterfullerenes" or "Bucky balls." [1, 4] Fullerenes can contain rings of five and six carbon atoms assembled like GPs, but also allow a much wider range of structures. Fullerenes can contain rings of size seven or more. Fullerenes typically contain non-planar rings, because the carbon atoms minimize energy functions based on angles and distances, and usually have no incentive to fall into planes. Furthermore, Fullerenes are not necessarily icosahedral and most have much less symmetry.

It is well known how the discoverers of C60 chose a name honoring Buckminster Fuller because they did not know enough geometry to realize the truncated icosahedron was already a well-known mathematical structure. [1] They alluded to geodesic domes for their lightness, strength, and the internal cavity, but I claim the name is one Fuller would disavow. Fuller was specifically interested only in triangulated structures. "If we want to have a structure, we have to have triangles." [5] He understood that triangulated structures remain rigid due to the lengths of the struts even if the angles at the joints are not fixed. So C60, like all GP having open hexagons and pentagons, is not even a "structure" by Fuller's definition.

122 Faces, GP(2,2). The next simplest of the family is GP(2,2), shown in Figure 15. Its derivation is best left for the method below, but It is a good example to introduce the notion of "paths." In Figure 0, we saw a kind of "knight's move" with *a* steps forward and *b* to the side, which underlies the GP(*a,b*) notation. Now imagine walking on a giant GP globe, taking one step per face, but always trying to go straight. When you step into a hexagon from a neighboring face, there is a natural opposite face to visit next, giving a sense of a straight walk, because hexagons have an even number of sides. If you step into a pentagon, which has an odd number of sides, there is no opposite face to exit to, so the path ends. An interesting puzzle is to predict where a path leads you if you continue walking straight.

Fig. 15 *GP(2,2) consists of twelve pentagons and 110 hexagons. It is reflexible. A pentagon-to-pentagon path and an all-hexagon round-the-world path are highlighted.*

Figure 15 shows that in GP(2,2) there are two types of paths. If you step off a pentagon and start walking straight, you end up at another pentagon after six steps. A second type of path goes once around the world, like an equator, with 18 steps in its cycle. Each hexagon is the crossroads for three paths. Seeing the paths helps one understand the structure. When holding a physical GP model, it is fascinating to follow the various types of gently meandering paths as you spin the ball in your hands, trying to make sense of where you end up from a given starting pentagon. Look back to GP(2,1) for an example where every path takes the traveler halfway around the world, from a pentagon to its antipodal pentagon. In GP(2,2), every face can be reached starting from at least one pentagon, but look at GP(3,0) for the simplest example where some of the hexagons can not be reached in a straight path if one always starts at a pentagon. GP(3,0) also illustrates parallel equatorial paths—you and a friend can walk together around the world side by side holding hands, each on your own 15 step cycle.

132 Faces, GP(3,1). This is another example which is most easily derived by the systematic generation methods below. The complexity of GP(3,1) is just enough that it is borderline between simple and intricate. Its chirality makes it an interesting subject for a sculpture. Figure 16 shows two wood versions I made using a laser-cutter to fabricate the 132 planar parts. To fit precisely, the parts must be made with high accuracy. Cutting the 120 irregular hexagons might drive one insane if using traditional tools such as a band saw. But a computer-controlled laser-cutter is ideal for producing irregular parts from flat material. My primary goal was the open-faced one, because Leonardo's solid-edge style lucidly presents an unfamiliar form with front and back simultaneously visible. The parts of the smaller, solid-faced one are the holes removed from the larger, open-faced one. For this to work out, one must be careful to design the openings to be proportional to the face shapes.

Fig. 16 *Wood sculptures, 8 inch and 6 inch diameters, of GP(3,1).*

492 Faces, GP(5,3) and GP(7,0). For the purposes of this introduction, skip now to 492 faces, to illustrate the smallest case where the number of faces does not determine the structure. It turns out that there are two topologically different forms of GP with 492 faces. Examining their paths can clarify their different structures. Each has only one type of pentagon-to-pentagon path, which is marked in Figure 17. In GP(7,0) these paths are of length seven, but in GP(5,3) the paths are of length 49 and go more than once around the world. While combinatorially distinct, both contain 492 faces, 980 vertices and 1470 edges.

Fig. 17 *Two GP forms each have 492 faces but their pentagon-to-pentagon paths have very different structures.*

2562 Faces, GP(16,0). As one final introductory example, Figure 18 shows the structure of GP(16,0). It serves to illustrate that one can generate GPs with an arbitrarily large number of faces. This particular example can be found in Buckminster Fuller's Expo67 dome in Montreal. If a large geodesic dome is designed where the skin is just a single layer of triangles, local regions of the surface are approximately a planar grid of triangles, and so can deform under local loads. A solution to this problem is to make a two-layer dome. One layer is a triangular tessellation and the other is its dual. Connecting the two layers with a network of triangles makes a rigid truss system. In the Expo67 dome, the inner layer is the hexagons and pentagons of GP(16,0) and the outer layer is the dual triangulated structure.

Fig. 18 *GP2562 has 12 pentagons and 2550 hexagons. It underlies the Expo67 dome in Montreal. Find the pentagons in this chicken wire!*

2. General Approach

Mathematics is the study of patterns. Michael Goldberg knew of various examples from the above list and generalized the underlying pattern to a systematic means of understanding and generating the complete family. Of his many interests, one was the problem of finding, for each *n*, which *n*-sided polyhedron is the most sphere-like, in having the highest volume to surface ratio. [9] In one paper he considered various particular polyhedra as candidates and proved that for n=12, the solution is the regular dodecahedron. [7] In a second paper, he followed up and defined the class of polyhedra considered here. [8] First consider just the topology of these polyhedra, without concern for the exact lengths and angles. In a second step, one can consider various ways to make planar-faced geometric realizations of the topology.

Goldberg's key insight is indicated in Figure 19. Using the vertices of triangular graph paper, one can generate a large equilateral triangle, ABC, by moving *a* steps to the right and then *b* steps to the upper right. Any integer pair (*a,b*) takes us from vertex A to some vertex B. Then that same motion, but rotated counterclockwise 120 degrees, takes us from B to C, and rotated once again takes us from C back to A. The three movements are congruent, so the large triangle, ABC, is equilateral yet has all three vertices on the triangular lattice. The example shows that the edges of ABC need not be parallel to the graph paper axes. If we take twenty copies of ABC and assemble them as an icosahedron, we get a continuous pattern of the small triangles. The result is like the k-frequency icosahedron of Figure 12, but allowing for a rotation between the triangular tessellation and the icosahedron faces. Small triangles at the edges of ABC can "fold" along an icosahedron edge and be part of two icosahedron faces. Extending the terminology "*k*-frequency" icosahedron, this can be called an "(*a*,*b*)-frequency icosahedron." The special case of *b*=0 gives the simplest, parallel type subdivision. Another special case, where *a*=*b*, makes the edges of ABC perpendicular to the edges of the small triangles and the result is again reflexible. The in-between values, where 0<*b*<*a*, give the chiral patterns.

Generating (*x,y,z*) coordinates for these vertices is a straightforward exercise in coordinate geometry. A linear transformation mapping ABC to an icosahedron face will map the small triangle grid points to our desired points in the icosahedral face planes. Each vertex is then projected to a unit sphere to give a triangulated geodesic sphere as in Figure 20. Many references on geodesic domes give the basic geometric ideas. [2, 6, 20, 24] For architectural purposes, the points may be redistributed slightly to reduce the number of different edge lengths, as that simplifies the physical construction if cutting metal struts. Details of one algorithm for the general (*a,b*) case are given in [14].

Figure 19. *An equilateral triangle, ABC, drawn on triangular graph paper. The oblique (a,b) coordinate system indicates how to move from A to B. Here (a,b)=(3,1).*

Figure 20. *Vertices of (3,1)-icosahedron in twenty icosahedral face planes and after projecting to unit sphere.*

Geometric Realizations. Given the above method for producing any (*a*,*b*)-frequency triangulated geodesic sphere, there are many ways to choose particular coordinates for the pentagon and hexagon vertices. Here are four methods:

Method 0, Vertex Elimination: One can simply leave out one third of the vertices (and their incident edges) to create a hexagon pattern that lies in the triangular grid. Formally, if the triangle vertices are indexed (*i,j*) in the natural way, meaning (0,0) is a 5-fold vertex and then take *i* steps to the right and *j* steps to the upper right, then simply omit the vertices where *i-j* is a multiple of 3. This leaves an open pentagon surrounded by open hexagons. If the the size of the grid is such that the neighboring 5-fold vertex is also omitted, i.e., if *a-b* is a multiple of 3, then this pattern matches with itself all over the sphere to give a grid of hexagons and pentagons. However, in general they are not planar. So this method is suitable for making a structure of edges, but it is not suitable for architectural purposes where one wants to cut flat sheets, e.g., plywood, to use as faces. It was used to produce the computer graphics in Figure 9, but it would not be suitable for making the wood sculptures shown in Figures 10 and 16, which have planar components.

Method 1, 3D Reciprocal construction: Given the vertices of the triangulated geodesic sphere, construct a plane tangent to the sphere at each point. Each plane defines a half-space outside the sphere and an "inside" half-space that includes the sphere. Consider the region inside all the planes, i.e., the intersection of all the inside half-spaces. By construction, its boundary faces will be planar hexagons and pentagons. Each vertex is easily located as the intersection of three planes by solving three simultaneous linear equations. (Technically, this method is taking the reciprocal in the sphere of the convex hull of the original vertices.)

Method 2, Centroid tangency: Do the above construction, but first redistribute the tangent points slightly on the unit sphere. Goldberg was interested in polyhedra that maximize the volume-to-surface-area ratio. There is a theorem that in the optimal solution each face will be tangent to the unit sphere at its centroid. [7] We would like to solve for the set of tangency points which have this property, but it gives a complex set of nonlinear equations to solve simultaneously, so we do not hope to solve it exactly in general. But one can find a numerical solution by a simple iterative algorithm. Start with an approximate solution for the tangency points, e.g., use Method 1. Find the face planes, calculate the centroid of each face, and project each centroid to the sphere to get a new set of points that can again be used to generate tangent planes. The optimal solution is a fixed point of this iteration. A simple computer program shows that iterating this process many times leads numerically to an approximation of the fixed point.

Method 3, Canonical Form: For any simple polyhedron, there is a canonical form in which all the edges are tangent to the unit sphere and the center of gravity of these tangency points is the origin. [26] This "Kobe-Thurston-Schramm-Conway canonical form" is of mathematical interest in part because the dual can also be made so its edges are tangent to the same sphere at the same points. There are various algorithms to calculate this canonical form, one of which is a simple fixed-point iteration analogous to that in Method 2. [13]

Note that Methods 1 and 2 give a polyhedron that is circumscribed around a sphere, meaning the faces are tangent to a sphere. Method 3 gives one which is "midscribed," meaning the edges are tangent to a sphere. Method 0 attempts to give an inscribed result, with vertices on the sphere, but that is not generally attainable with planar faces. The dodecahedron and truncated icosahedron are inscribable, but it is not clear whether or how the more complex GPs are inscribable while preserving symmetry. That is left as an open problem.

After implementing these methods, I find that the simple Method 1 is suitable for most sculptural design purposes that motivate me. Figure 21 shows GP(3,1) as realized geometrically in three forms. In the optimal volume-to-area form (Method 2) the hexagons adjacent to pentagons have a larger variation in their edge lengths than I prefer. In the canonical form (Method 3) the pentagons are smaller than I prefer. So on purely subjective grounds I am happy to use Method 1, which incidentally has the advantage of being fastest to compute. But for other purposes, other forms may be more suitable.

Figure 21. *Three geometric realizations of GP(3,1) using methods 1, 2, 3 respectively.*

Counting Components. It is often useful to know how many faces, edges, and vertices there are in GP(a,b). For example, one needs to know how many pieces are required before making a physical model. First consider its dual, the triangulated sphere (a,b) as in Figure 20. In terms of Figure 19, moving one unit in the a direction is equivalent to moving (1,0) in XY coordinates and moving one unit in the b direction is (1/2, sqrt(3)/2) in XY coordinates. So the Pythagorean theorem and a bit of algebra show that the total movement from A to B has length squared equal to a^2+ab+b^2. This formula is central to what follows, so define $d = a^2+ab+b^2$. The area argument in the discussion of GP(3,0) shows that there will be d small triangles for each icosahedron face. As an icosahedron has 20 faces, there are $20d$ small triangles in a complete sphere such as Figure 20. Imagine cutting those $20d$ small triangles out of paper to assemble them. There are three edges each, so we would have $60d$ small triangle edges in a pile of loose paper triangles. After taping the edges together in pairs to make a geodesic sphere, there would be $30d$ edges in the polyhedron. Euler's theorem tells us that V+F=E+2 in any simple polyhedron, so knowing F and E, we can solve for V to determine there are $10d+2$ vertices in the triangulated geodesic sphere.

GP(a,b) is the dual to the triangulated sphere. The effect of this is to swap the values of V and F because each original face generates one vertex of the dual and each original vertex becomes surrounded by a face of the dual. The number of edges is unchanged because each edge of the dual crosses exactly one of the edges of the original. So the parts counts for GP(a,b) are given by:

Define $d = a^2+ab+b^2$.

Number of Faces: $10d+2$
Number of Vertices: $20d$
Number of Edges: $30d$

As a and b are integers, so is d, and the formula $10d+2$ explains why the number of faces is always 2 more than a multiple of 10. As to the mysterious sequence 12, 32, 42, 72, 92, 122… these are all the values obtainable when using a positive value for a and a non-negative value for b. The smallest case where there are two sets of (a,b) values that give the same d are (5,3) and (7,0) which both give 492 faces. Goldberg gives a construction showing that there are larger examples with not just two, but any desired number of different forms resulting in the same number of faces. [8]

Paths. Straight paths of hexagons correspond to the edges of the small triangles in the underlying lattice. Because they are highly structured, yet we can only see a small part of them at once as they disappear over the horizon, they provide a pleasant sense of mystery.

Analysis of Figure 19 shows that the length of a pentagon-to-pentagon path in GP(a,0) is a and in GP(a,a) is $3a$. In the chiral cases, the length is d in GP(a,b) if a and b are relatively prime. More generally one must divide out the greatest common divisor, so the path length is d/GCD(a,b) in the chiral cases. The length of the equatorial paths is $5a$ in GP(a,0) and $9a$ in GP(a,a). In the chiral cases where a and b are relatively prime, there are no equatorial paths. But the chiral cases with GCD(a,b) > 1 are mysterious; "equatorial" paths can wrap around several times, crossing over themselves repeatedly. I can write a program to build the structure and count the answer, but a simple analysis and a simple formula for the length(s) is left as an open problem.

The question of where a finite path ends up does not seem to have an easy answer in general. Starting at a pentagon, there are three possibilities: the five paths may end at the five near neighboring pentagons, they may end at the five medium distance pentagons, or they may all end at the one opposite pentagon. No path can end at the pentagon is starts from. (One way to see this is because paths are reversible, such a loop would violate the 5-fold symmetry about that vertex; would it lead clockwise or counterclockwise?) When b=0, paths end at a near neighbor and if a=b, paths end at a further neighbor. But the chiral cases can be subtle. Paths across hexagons correspond to lines of the small triangle lattice. It is easy to follow them and see where they meet a vertex of the large triangle lattice. For example, if $a=2b$ or if $2a=7b$, the path can be traced to end at the opposite pentagon. The difficulty is in understanding how long paths wrap multiple times around the icosahedron when it is unfolded. I leave it as an open problem to give a simple method of determining for which GP(a,b) the paths end at the opposite vertex.

A related open question is to give a formula for determining how many hexagons can not be reached on any path that starts at a pentagon. Here again, the reflexible cases are straightforward. In the chiral case, the answer is easy when a and b are relatively prime, as then every hexagon can be reached. But the remaining cases are left as another open problem.

Going Further. This paper focuses on GPs only, but it can be mentioned that there are many natural generalizations of these polyhedra. One might allow seven-sided faces, tetravalent vertices, and so forth, or consider polyhedra without icosahedral symmetry, to explore many wider sets of polyhedral forms. One interesting theorem is that even if the symmetry requirement is relaxed, in any trivalent polyhedron with only pentagonal and hexagonal faces, there are always exactly twelve pentagons and there are examples with any number of hexagons other than one. [10] But there is a certain naturalness that draws me to stay within the GP family. It is certainly sufficient to provide inspiration for many novel constructions, as the following section illustrates.

3. Models and Artwork

The idea of making paper models was promoted in Section 1. They are very cost-effective means for gaining 3D understanding in a hands-on manner. The model in Figure 5 was expediently assembled using tape on the inside. For a more elegant, tape-free model, one can cut out the faces with tabs around the edges and then fold back the tabs and glue them internally. In the “one-tab” method, at each edge just one of the two faces has a tab, which is glued to the inside of the other face. In the “two-tab” method, all edges of all faces have a tab and they are glued together in pairs to make a rib that lies under each edge. I find this two-tab method to be faster, easier, and more attractive. More complex variations can be made with the methods in [25].

Scissors are adequate, but the work of cutting paper parts is greatly simplified if one has a robotic paper cutter. The example in Figure 5 was made with the aid of an inexpensive computer-controlled paper cutter. These are now marketed for home use and are sufficiently affordable (under $200) that any serious model maker should consider owning one. They allow one to easily make models which would be far too tedious to cut with scissors. [16] I expect such cutters will protect many future paper model aficionados from Carpel Tunnel Scissors Syndrome.

Paper also has the advantage that one can easily print on it. For example, patterns can be found online for globes of the Earth projected onto various polyhedra. The most complex GP that I have seen used as a globe of the earth is the truncated icosahedron, which was first used by John Snyder in 1992 as the basis of an equal-area projection [22]. It is mathematically straightforward to project geographic data to the face planes of any polyhedron. More complex GPs would naturally lead to better representations of the sphere.

Wooden examples of GP were shown in Figures 10 and 16. They are more difficult to make than it may appear. Similar models can also be made with acrylic or other plastics. The parts must be carefully cut if they are to fit without gaps. The edges must be beveled accurately to the correct dihedral angles. While laser-cutters and other computer controlled technologies can cut out the shapes, current machinery does not easily bevel the edges. So this operation must be done manually and care must be taken not to flip chiral parts or mix up the many different angles in one model.

A second approach to wood models, especially for large ones, is to build up pentagons and hexagons from individual mitered pieces, one for each polygon edge. This is the technique I used for the 18 inch GP(1,1) model shown in Figure 22. It is intended as a reconstruction of the wood model that Leonardo drew, when he made the drawing shown in Figure 3. Each pentagon is made from five mitered edge pieces and each hexagon from six pieces. The faces’ edges were then beveled to the proper angles and glued together.

Figure 22. *Reconstruction of Leonardo’s GP(1,1) model. Cherry, 18 inches. Compare to Figure 3.*

Exact models of only the simplest GPs can be made with familiar mathematical kits such as Zometool or Polydron. While these kits are excellent for other families of polyhedra, they each have only a certain inventory of lengths and angles built into them, but GPs require many different lengths and angles. A flexible-edge construction set, of the type used by chemistry students for making molecular models, is excellent for making GP models. In these kits, there are rigid trivalent connectors to use at each vertex and slightly flexible tubing to use for edges. The tubing can be cut to make any desired edge lengths. Figure 23 shows a 50 inch model of GP(5,1) which I made with the Stony Brook University Chemistry Club. The model is an approximation, because the trivalent connectors have all 120 degree angles, because the edges are slightly curved, and because the polygons in the model are not necessarily planar. In addition, for ease of construction, I chose just two lengths for the edges, choosing their ratio so as to give as spherical an approximation as possible within the limits of that constraint. It was an educational group project to assemble it and hang it for display in the Chemistry library. Instructions are available online at [18].

Figure 23. *50 inch model of GP(5,1) assembled by the Stony Brook University Chemistry Club.*

Many geometric constraints are removed when one uses additive fabrication technology to make physical models. These technologies, sometimes called rapid prototyping, solid freeform fabrication, or 3D printing, can robotically build any describable form. They are ideal for making GP models, with their many slightly different lengths and angles. Figure 24 shows two models of GP(8,3) made using the same geometry description file on two different fabrication machines, one in plastic and one in metal.

Figure 24. *Two models of GP(8,3). Left: 5-inch. epoxy polymer. Right: 2.5 inch, steel and bronze.*

There is a long tradition of turning nested ivory spheres on a lathe, where each inner sphere is carved with a tool that is passed through the holes of the outer layers. The tradition started in Europe in the sixteenth century, but in recent centuries has been associated more with Japan and China. Modern examples can be purchased made of carved plastic instead of ivory. In homage to that tradition, I designed a set of nested GP forms which can be produced on additive fabrication machines. Figure 25 shows a set of ten different concentric forms: GPs (5,0), (3,3), (4,2), (5,1), (6,0), (4,3), (5,2), (6,1), (4,4), and (5,3), which steadily increase from 252 to 492 faces. In the traditional nested ivory spheres, the pattern of holes in each sphere is identical, because tools must pass through holes in the outer layers to cut into the inner layers. In this modern variation, there is no similar constraint and each layer can be designed independently.

Figure 25. *Ten different nested GP forms, from GP(5,0) to GP(5,3) and cut-away rendering. The outer one is 3 inches in diameter.*

When one first studies the Platonic and Archimedean solids, the snub cube and the snub dodecahedron stand out for their mysterious chirality. The term "snub" refers to a chiral way of separating the faces with a border of triangles. If we apply this same chiral snubbing operation to the truncated icosahedron, we can generate two forms of the "snub truncated icosahedron". A model of one of its two enantiomorphs is shown in Figure 26. One way to generate it starts with the dual to GP(4,1), which includes these triangles plus five triangles in each pentagon and six triangles in each hexagon. Simply delete the vertices and edges incident to the 3-fold and 5-fold axes. This is like the vertex deletion method #0 of Section 2, except I specifically leave many of the triangles in place to be part of the final result.

Figure 26. *Snub truncated icosahedron, derived from dual to GP(4,1) by deleting selected edges. 3 inch rapid prototyping model.*

More interestingly, when we start with a chiral polyhedron, e.g., GB(2,1), which already has two mirror-image forms, and apply the snub operation, there are four possible results. Figure 27 shows the four possible forms of the "snub GB(2,1)." Note that two of them are based on the (7,0) triangulation and the other two on (5,3), which we saw above give the same number of components but differently arranged. I have not yet made physical sculptures based on these four forms, but I plan to, as they are visually interesting in the way they combine a triangulated structure with openness, and intellectually interesting for illustrating two levels of chirality to ponder.

Figure 27. *Four different forms of snub GB(2,1) illustrate two levels of chirality.*

If one applies an affine transformation to a GP, the result is an ellipsoid-like form which still has planar faces. For architectural purposes, an oblate form might be used to cover a wide area with a dome that need not be as high as a spherical dome of the same footprint. Figure 28 shows an oblate design of this form, but not with architectural intent. It is designed as a decorative bowl. The form began as GP(5,3), one of the two GPs with 492 faces. I compressed it along a 5-fold axis, and removed struts around one pole to create a large pentagonal opening for the top. I considered flattening the bottom slightly so it sits steadily on a table, but then decided to leave the curve so it can rock dynamically. If I get a chance to make a matching dish, I plan to perform the same transformations, but start with the other version, GP(7,0).

Figure 28. *Candy dish, 5 inches, nylon, based on oblate GP(5,3)*

The problem of designing an elegant, mathematically described egg goes way back, at least to D'arcy Thompson's classic *On Growth and Form*. (Ron Resch constructed a very nice triangulated solution in his giant Ukranian Egg sculpture.) Figure 29 shows a simple idea based on affine transformations. The form started as GP(7,2) but the North and South hemispheres were each given an independent affine transformation. One half was stretched and the other compressed. (So most faces remain planar, but those that cross the equator have a slight crease.) In addition, I rounded the openings to be smoother and more circular instead of the original hexagons and pentagons. The result has a very satisfactory sense of lightness and fragility, clearly an egg yet somehow alien. I especially enjoy the fact that all the paths sesquicircumscribe the egg—they rotate exactly 1.5 revolutions to end up at the antipodal pentagon.

Figure 29. *Perforated egg sculptures, 3 inches, nylon, based on GP(7,2)*

Figure 30 shows an idea which grew as a development of the puzzle in Figure 9. It is a spherical jigsaw puzzle which assembles to make GP(5,3). It is divided into twelve identical parts along edges in such a manner that each piece is centered on a pentagon and has 5-fold symmetry. This is possible for all GP in which the number of faces is a multiple of 12. Because these parts are identical, it is an easy puzzle to solve, but still quite fun to play with. This initial example was intended in part to test the design idea and verify that the parts will hold together by friction alone. Future variations on the concept will incorporate different shapes for the pieces or an egg-shaped solution, to provide more a more challenging puzzle.

Figure 30. *Twelve-part assembly Puzzle, 5 inches, nylon, based on GP(5,3)*

The discussion of GP(16,0) explained that two-layer geodesic domes are much more resistant to local deformations than single-layer domes. The Expo67 dome mentioned there and all other two-layer domes that I have seen are based on reflexible structures. The same principles apply to chiral forms, but they are more costly to build because a larger inventory of parts lengths is required. As I find the chiral forms to be visually engaging, I designed software to create two-layer domes based on any GP form. Figure 31 shows what I believe is the only chiral two-layer dome in existence. The inner layer is GP(3,1) and the outer layer is its dual. Educational, physical models of two-layer spheres can be made with flexible angle kits as shown in [19]. While the flexible angles would not make a rigid model of a GP alone, when triangulated with its dual, it becomes rigid.

Figure 31. *Two-layer geodesic sphere based on GP(3,1), 4 inches, nylon. As far as I know, it is the world's only chiral two-layer geodesic structure.*

As a final example, the sculpture shown in Figure 32 is a kind of chain mail sphere. Starting with GP(7,4), the faces were replaced with small circular rings. Each edge of the dual triangulated structure was replaced with a link that connects adjacent rings. The dimensions of the rings and links were chosen so they do not overlap each other. The resulting chain mail fabric is flexible, as can be seen in the close-up image. If every link were free to move, the whole form would collapse like an empty sack. So to give the overall sculpture a shape, a subset of the links were modified to lock with their rings, making a rigid skeletal form corresponding to the edges of a dodecahedron. Twelve circular regions of the mesh are free to move within this rigid framework. The total effect is quite remarkable and indicates how many fascinating possibilities might be encountered when one explores a rich subject such as the Goldberg Polyhedra.

Figure 32. Sculpture with chain links based on GP(7,4), 6 inches, nylon.

GP Conclusions

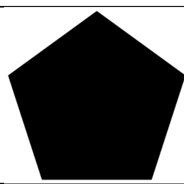
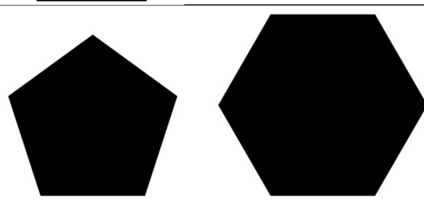
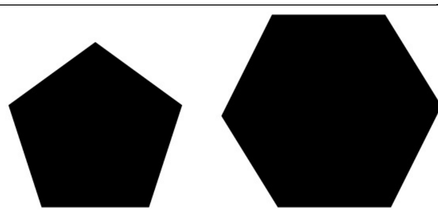
Goldberg polyhedra gather praise for a gamut of properties. They are girdled in polygons making gyrating paths in which we glimpse open problems. I hope their gorgeous patterns grow popular among gluers of paper. If your goal is play, these globes provide geometric puzzles. It is my genuine pleasure to grandly present this gallery of possibilities.

Appendix

a	*b*	$d=a^2+ab+b^2$	**Faces**	**Vertices**	**Edges**
1	0	1	12	20	30
1	1	3	32	60	90
2	0	4	42	80	120
2	1	7	72	140	210
3	0	9	92	180	270
2	2	12	122	240	360
3	1	13	132	260	390
4	0	16	162	320	480
3	2	19	192	380	570
4	1	21	212	420	630
5	0	25	252	500	750
3	3	27	272	540	810
4	2	28	282	560	840
5	1	31	312	620	930
6	0	36	362	720	1080
4	3	37	372	740	1110
5	2	39	392	780	1170
6	1	43	432	860	1290
4	4	48	482	960	1440
5	3	49	492	980	1470
7	0	49	492	980	1470

Table 1: GP(*a,b*) data for all cases under 500 faces.

For each *a* and *b*, Table 1 gives value of *d*, and number of faces, vertices, and edges, sorted by *d*.

GP(1,0) [12]	
GP(1,1) [12, 20]	
GP(2,0) [12, 30]	

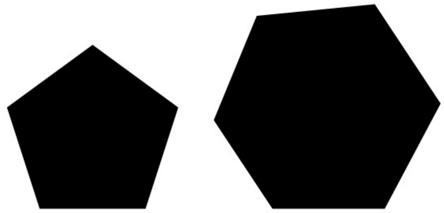
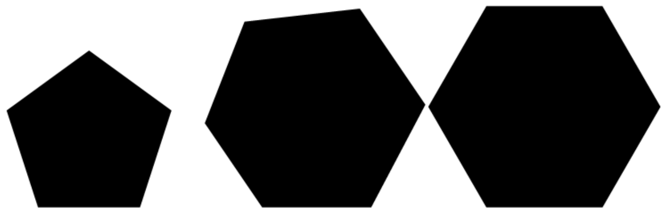
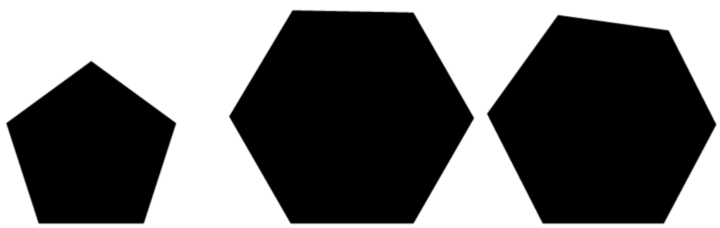
GP(2,1) [12, 60]	
GP(3,0) [12, 60, 20]	
GP(2,2) [12, 60, 30, 20]	
GP(3,1) [12, 60, 60]	

Table 2: Face shapes for making models of the first seven GP.

To make paper models, enlarge the templates in Table 2, cut out the indicated number of copies of each part [shown in braces] and tape them together. Small faces are difficult to work with, so it is easiest if the scale for enlargement makes the edges at least 2 cm long. Be organized about the hexagons, so as not to mix them up or turn them over. For pentagons, 12 pieces are needed; for hexagons, 20, 30, or 60 are needed of each shape, according to whether the hexagon is on a 3-fold axis, a 2-fold axis, or no axis respectively. The position in terms of symmetry axes and matching edge lengths give enough information to connect the hexagons correctly. These shapes are for the canonical form described as Method 3 of Section 2.

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